A database of polarised K3 surfaces

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Abstract

We describe a computer-based database of polarised K3 surfaces and explain the meaning of the information it contains. In a precise sense, the database includes all K3 surfaces.

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1 Introduction

Many authors have compiled lists of K3 surfaces embedded in weighted projective space (wps). The first of these lists is the 'famous 95' weighted hypersurfaces of Reid [1980] (Theorem 4.5) and others. Next there are 84 families of K3 surfaces in codimension 2 computed by Fletcher [2000] (section 13.8), followed by 70 families in codimension 3 and 142 families in codimension 4 both computed by Altınok [1998]. Such lists could be continued indefinitely in increasing codimension, as there are countably many deformation families of polarised K3 surfaces, although the construction of explicit equations becomes difficult.

We extend the classification of polarised K3 surfaces to give a list that contains the numerical data of all polarised K3 surfaces in the precise sense of Theorem 8 below. Although the list of families of polarised K3 surfaces is infinite, the numerical data we work with behave in a regular way after the first 15,000 or so families are obtained, and so a finite list can summarise the whole classification. Even so, it is far too large to be reproduced in the way that the existing lists have been. In fact, both the analysis used to create the list and methods of interrogating it are handled by a computer. The resulting list of 24,099 numerical K3 candidates (see Definition 6) is known as *the K3 database*. It was created using the computer algebra system Magma [Cannon, 2005; Bosma *et al.*, 1997], and it is accessible in three ways: one can run Magma itself, or connect to the web interface at [Brown *et al.*, 2004] (which runs Magma in the background) or install a SQL-style database [Brown and Kerber, 2005] prepared from the on-line version. These are discussed in section 4. Although computer access is the only serious way to address such a database, K3 surfaces in low codimension are also available at [Brown *et al.*, 2004] including a new list of 163 K3 surfaces in codimension 5.

The main results of this paper explain the meaning of the K3 database. We make this explicit in Meanings 7, 9, 10, 18. Theorem 8 explains the sense in which it is comprehensive and the way in which we regard the K3 database as an upper bound for the numerical data of polarised K3 surfaces. An immediate corollary is a sharp lower bound on the degree of polarised K3 surfaces; see section 4 for Magma code that makes this calculation.

Computation 1 *If X, A is a polarised K3 surface, then the degree A^2 of X is at least $1/330$. In more detail, both the degree A^2 and the Picard number $\rho_X \leq 20$ of X are bounded below according to the genus $g = h^0(X, A) - 1$ as* *follows:*

g	-1	0	1	≥ 2
lower bound for A^2	$1/330$	$1/42$	$1/2$	$2g-2$
lower bound for ρ_X	10	6	2	1

We clear up two points of confusion at once. First, there is no claim that every numerical candidate in the K3 database comes from a polarised K3 surface. Indeed, in Theorem 12 we show that one particular candidate does not to arise as a K3 surface, at least not in an easy way. Second, while each candidate in the database is given a plausible description as a K3 surface embedded in wps, there is no claim that this this description can be realised, even when there is a K3 surface whose invariants match those of the candidate.

More positively, there are various ways in which a candidate in the database may be justified. One is to write down equations for a K3 surface in wps and confirm its properties. This is done for all candidates in codimensions 1, 2 and 3 in [Iano-Fletcher, 2000] and [Altınok, 1998]. Another is by unprojection which is discussed in section 3; this is a 'bottom up' approach, constructing complicated surfaces from easy ones. The reason for discussing it here is not to propose to carry out the unprojection calculations but to explain the descriptions of candidates in the database.

For the rest of this introduction, we explain the purpose of this classification and relate it to others in the literature.

There are many reasons for assembling reasonably large databases of varieties rather than restricting attention to those instances of classification that result in short lists. Belcastro [2002], for instance, uses the famous 95 as a testing ground for K3 mirror symmetry. Johnson and Kollár [2001] construct and use lists of weighted hypersurfaces to find varieties admitting a Kähler–Einstein metric. Corti, Pukhlikov and Reid [2000] use the famous 95 as the starting point for a systematic and explicit study of birational rigidity and the Sarkisov Program for Fano 3-folds. In some of these cases, one could regard lists of varieties as being merely a convenient source of many examples, rather than an essential ingredient. But already Belcastro is hampered by restricting to hypersurfaces, since, not surprisingly, in seeking mirror pairs she finds hypersurfaces whose partner, if it exists, is not another hypersurface in wps.

The main reason for extending the lists as we do is as part of the classification of Fano 3-folds. We explain this briefly; see [Altınok *et al.*, 2002] for much greater detail. A 3-fold X is a *Fano 3-fold* if it has at worst $\mathbb{Q}$-factorial terminal singularities and $-K_X$ is ample—it is common to insist that moreover $\mathrm{Pic}(X) = \mathbb{Z}$ and $-K_X$ is a generator. By [Kawamata, 1992], there are only finitely many deformation families of Fano 3-folds. If the linear system $|-K_X|$ contains an irreducible surface S with only Du Val singularities, then S is a K3 surface and it is polarised by the trace of $-K_X$. Such a surface S is called a *K3 elephant* for X, and the vast majority of known Fano 3-folds have a K3 elephant. The main point of [Altınok *et al.*, 2002] is to attempt the converse operation: given a polarised K3 surface S, A, construct a Fano 3-fold X having S as its K3 elephant. This can be regarded as a deformation–extension problem, in which one must include a new variable in the equations of S while maintaining the irreducibility (at the very least) of the locus they define. From this point of view, the K3 database contains a coarse classification of Fano-with-elephant 3-folds as a finite sublist (although exactly which sublist is the interesting point).

There are many other lists of varieties we could mention. Following classifications by Miranda and Persson [1989] and others, Shimada and Zhang [2001], [Shimada, 2000] classify K3 surfaces that arise as elliptic fibrations. Kreuzer and Skarke [1998] classify K3 surfaces that arise as toric hypersurfaces, and in higher dimension, they classify Calabi–Yau 3-fold toric hypersurfaces [Kreuzer and Skarke, 2000]. Their famous Calabi–Yau database contains nearly 500 million families of Calabi–Yau 3-folds; it is not known whether there are infinitely many families or not. Buckley and Szendrői [2005], [Buckley, 2003] construct Calabi–Yau 3-folds by methods similar to those we use here, although their interest is not in constructing lists of varieties but rather to find examples not already in the vast Kreuzer–Skarke list. More recently, Caravantes [2005] computes examples of Fano 3-folds that are quotients of other Fano 3-folds—so-called Fano–Enriques 3-folds—in codimension at most 3, and Kasprzyk [2003; 2005] computes the classifications of toric Fano 3-folds under various hypotheses.

2 Families of K3 surfaces

The methods used here follow Altınok's approach using Hilbert series, as explained in Altınok–Brown–Reid [2002].

2.1 Polarised K3 surfaces

A *polarised K3 surface* is a pair X, A where X is a surface having only Du Val singularities, trivial canonical divisor $K_X = 0$ and irregularity $q = 0$, while A is an ample divisor on X. Recall that a *Du Val singularity* (or Kleinian or ADE singularity) is the germ at the origin of $\mathbb{C}^2/G$ where $G \subset \mathrm{SL}(2, \mathbb{C})$ is a finite group; equivalent conditions, see [Durfee, 1979] or [Reid, 1980], include being defined by an equation from the list of ADE normal forms, or imposing no conditions on adjuction so that the canonical class pulls back to a minimal resolution. Throughout this paper, *K3 surface* refers to such a pair X, A.

Graded ring of a K3 surface A polarised K3 surface has a graded ring $R(X, A) = \oplus_{n \geq 0} H^0(X, nA)$, and the *Hilbert series* $P_X(t)$ *of* X, A is defined to be the Hilbert series of this graded ring:

$$P_X(t) = \sum_{t \geq 0} h^0(X, nA) t^n.$$

Since A is ample, $R(X, A)$ is a finitely-generated k-algebra and the Proj correspondence embeds X in wps:

$$X = \operatorname{Proj} R(X, A) \subset \mathbb{P}^N \quad \text{for some} \quad \mathbb{P}^N = \mathbb{P}(a_0, \ldots, a_N)$$

where we suppose that $R(X, A)$ is minimally generated as a k-algebra by homogeneous elements $x_0, \ldots, x_N \in R(X, A)$ with $\deg x_i = a_i$. A minimal free resolution of $R(X, A)$ as a $k[\mathbb{P}^N]$-module then exhibits a preferred rational form of the formal power series $P_X(t)$:

$$P_X(t) = \frac{H_X(t)}{\prod (1 - t^{a_i})} \tag{1}$$

where $H_X(t)$ is a polynomial, the *Hilbert numerator* of X, A and the denominator product is taken over the weights $a_0, \ldots, a_N$ of the wps $\mathbb{P}^N$. The *codimension of* X, A is defined to be the codimension of X in this embedding. The *genus of* X, A is $h^0(X, A) - 1$, which is an integer ≥ -1.

Riemann–Roch and baskets of singularities Altınok's Riemann–Roch formula, Theorem 2 below, computes the Hilbert series of a K3 surface X, A. It involves the notion of a *basket of quotient singularities*, which is explained below, to compute the effect of the singularities of X, A on $h^0(X, nA)$.

Theorem 2 (Altınok[1998] Theorem 4.6, [2003] 3.2) *Let X, A be a polarised K3 surface. Then*

$$P_X(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{(1-t)^3}\frac{A^2}{2} - \sum_{\mathcal{B}} \frac{1}{(1-t^r)} \sum_{i=1}^{r-1} \frac{\overline{bi}(r-\overline{bi})t^i}{2r} \tag{2}$$

where

$$A^2 = 2g - 2 + \sum_{\mathcal{B}} \frac{b(r-b)}{r}. \tag{3}$$

In these formulas, $\mathcal{B}$ is a collection of cyclic quotient singularities $\frac{1}{r}(a,-a)$ at which the polarising divisor A restricts to the eigensheaf $\mathcal{L}_a$ of the quotient. The notation $\overline{x}$ denotes the minimal nonnegative residue of x modulo r, and $b = \overline{b}$ satisfies $\overline{ab} = 1$.

The collection $\mathcal{B}$ of cyclic quotient singularities is called the *basket of singularities of* X, A. It computes the contribution of the actual singularities of X, A to Riemann–Roch. In general the singularities of X may differ from $\mathcal{B}$. This phenomenon is well-known since [Reid, 1980] and [Reid, 1987], although here we need to know how baskets arise.

If $p \in X$ is a Du Val singularity, then it is also polarised by A—this global polarising divisor restricts to some element of the local class group of $p \in X$. Taking a small analytic neighbourhood $p \in U \subset X$, there is a deformation of $U, A_{|U}$ so that the general fibre U_t, A_t has only cyclic quotient singularities and at each such $q \in U_t$ the divisor A_t restricts to a generator of the local class group. Thus $q \in U_t$ is of type $\frac{1}{r}(a,-a)$ for coprime $0 < a < r$. Let $\mathcal{B}_p$ be the collection of these polarised cyclic quotient singularities. This collection $\mathcal{B}_p$ is uniquely determined by the polarised singularity $p \in X$. Finally, $\mathcal{B}$ is the collection of all $\mathcal{B}_p$ as p runs through the Du Val singularities of X. The following result is implicit in [Reid, 1987] (9.4).

Lemma 3 *In the notation above, let Γ_p be the dual graph of the resolution of $p \in X$ and $\Gamma_{q_1}, \ldots, \Gamma_{q_k}$ be those of the basket $\mathcal{B}_p$. Then the disjoint union $\cup\Gamma_{q_i}$ embeds as a subgraph of Γ_p so that no two components, Γ_{q_i} and Γ_{q_j} for $i \neq j$, are joined by an edge of Γ_p.*

Proof By [Reid, 1987] (9.4) and (4.10) (suitably re-ordered), the only polarised Du Val singularities that lead to a non-empty basket are:

$p \in X$	basket
A_{nk-1}	$k \times A_{n-1}$
D_{2k-1}	$(2k+1) \times A_1$
D_{k+2}	$2 \times A_1$
E_6	$2 \times A_2$
D_{2k}	$k \times A_1$
E_7	$3 \times A_1$.

(4)

In each case, the dual graphs of the basket can be arranged as a disconnected subgraph of Γ_p as claimed. Q.E.D.

It would be convenient to know that X deforms to a K3 surface with singularities equal to the basket, but we do not know that or need it.

Proposition 4 *If $\mathcal{B}$ is a basket for a K3 surface of genus g, then*

$$\sum_{\mathcal{B}}(r-1) \leq 19 \quad \textit{and} \quad 2g-2+\sum_{\mathcal{B}}\frac{b(r-b)}{r} > 0.$$

Furthermore, if the singularities of $\mathcal{B}$ lie on a surface Y, then the minimal resolution of these singularities must not contain 17 disjoint -2-curves and all coefficients of the power series $P(t)$ computed by formula (2) are non negative.

Proof If the singularities of X are equal to those of the basket (as polarised singularities), then all the claims are standard: the first comes from the bound on the Picard rank of the resolution; the second is $A^2 > 0$; the third is a standard consequence of the Torelli theorem. Even if the singularities of X are not those of $\mathcal{B}$, the second inequality holds automatically since the basket computes A^2 exactly.

In general, Lemma 3 shows that the number of exceptional curves in a resolution of X is at least that in the resolution of its basket, and moreover if one can find k disjoint -2-curves in the resolution of the basket then the same is true for the resolution of singularities of X itself. Q.E.D.

Computation 5 *Let B_g be the set of baskets which appear in Riemann–Roch for a K3 surface with genus $g \geq -1$. Then B_g is finite and of size*

g	-1	0	1	≥ 2
$\#B_g$	4281	6479	6627	6628

and moreover $B_g = B_2$ whenever $g \geq 3$.

When $g \leq 2$, this is the result of a simple computer enumeration of all possible baskets of singularities of type $\frac{1}{r}(a, -a)$ for coprime $0 < a < r$ satisfying the four conditions of Proposition 4. The fact that $B_g = B_2$ when $g \geq 3$ is immediate from the form of the degree condition $A^2 > 0$. The 'missing' basket in genus 1 is the empty one: there is no nonsingular K3 surface with $g = 1$.

2.2 The meaning of the K3 database

The K3 database is intended to represent all possible K3 surfaces X, A. Here we say in what sense every K3 surface appears in the database, and conversely we begin to see to what extent items in the database come from K3 surfaces.

Definition 6 *A* numerical K3 candidate *is a pair $(g, \mathcal{B})$, where $g \geq -1$ is an integer and $\mathcal{B}$ is a basket from the set B_g constructed in Computation 5.*

A numerical K3 candidate contains exactly the data needed to compute a Hilbert series using the formula of Theorem 2.

Meaning 7 *The K3 database is a finite set $\mathcal{D}_{\mathrm{K3}}$ whose elements are numerical K3 candidates. It includes the candidates $(g, \mathcal{B})$ for $-1 \leq g \leq 2$ and all $\mathcal{B} \in B_g$.*

For each candidate $\xi = (g, \mathcal{B})$, we define formally a degree denoted A^2_ξ and a Hilbert series denoted $P_\xi(t)$ by the formulas (3) and (2) respectively.

Theorem 8 (Completeness of the K3 database) *Let X, A be a polarised K3 surface of genus g. Then X, A is represented in the K3 database $\mathcal{D}_{\mathrm{K3}}$ as follows:*

- *if $g \leq 2$, then there is a numerical K3 candidate $\xi = (g, \mathcal{B}) \in \mathcal{D}_{\mathrm{K3}}$ with*

$$A^2 = A^2_\xi \quad \text{and} \quad P_X(t) = P_\xi(t).$$

- *if* $g \geq 3$*, then there is a numerical K3 candidate* $\xi = (2, \mathcal{B}) \in \mathcal{D}_{\text{K3}}$ *with*

$$A^2 = A_\xi^2 + 2(g-2) \quad \textit{and} \quad P_X(t) = P_\xi(t) + \frac{t(1+t)}{(1-t)^3}(g-2).$$

Proof If $g \leq 2$, then this follows immediately from Theorem 2 and Proposition 4. When $g \geq 3$, it holds because $B_2 = B_g$ from Computation 5 implies that the formulas (3) and (2) differ from the $g = 2$ case only by the $2g$ term in A^2.

Q.E.D.

Weights and codimension The K3 database includes extra information about each entry: each $\xi \in \mathcal{D}_{\text{K3}}$ has a sequence of *weights* $(a_0, \ldots, a_N)$ associated to it with positive integers a_i. The hope is that a K3 surface X, A exists with numerical data ξ and embedded by (all multiples of) A in $\mathbb{P}^N(a_0, \ldots, a_N)$.

The naive method to generate such weights generalises the first examples such as in [Altınok *et al.*, 2002] section 1; variations of it are described in [Iano-Fletcher, 2000] (section 18). If $P_\xi(t) = 1 + p_1 t + p_2 t^2 + \cdots$ is the Hilbert series of some ring R, then R must have p_1 generators in degree 1. We compute $(1-t)^{p_1} P_\xi = 1 + p'_k t^k + \cdots$ where p'_k is the first nontrivial coefficient. If $p'_k > 0$, then R must also have p'_k generators in degree k; in that case we compute $(1-t)^{p_1}(1-t^k)^{p'_k} P_\xi$ and continue. If $p'_k < 0$, then R must have at least $|p'_k|$ relations in degree k; in that case we stop the calculation and let the weights of ξ be the collection of weights of all generators deduced so far.

Thus, just as in section 2.1, the weights determine a preferred rational expression for the corresponding Hilbert series. Its numerator is again called the *Hilbert numerator* and is denoted $H_\xi(t)$ in this context. (In principle, it is possible that the calculation breaks down too soon and H_ξ is not a polynomial, but in practice this does not happen.) In this way, the weights of ξ determine a prediction of a K3 surface $X \subset \mathbb{P}^N(a_0, \ldots, a_N)$ that realises ξ. With this in mind, the *codimension* of ξ is defined to be $N - 2$.

Meaning 9 *The K3 database* $\mathcal{D}_{\text{K3}}$ *comprises all pairs* $(g, \mathcal{B})$ *with* $g \leq 2$ *and* $\mathcal{B} \in B_g$ *together with those pairs with* $3 \leq g \leq 9$ *having codimension at most 7. For each genus* g*, the pairs* $(g, \mathcal{B})$ *are listed in increasing order of Hilbert series.*

The order on Hilbert series is of course the natural lexicographic order. The weights, and hence the codimension, are those computed in section 3 using more systematic methods than the naive one above. The number of numerical K3 candidates per genus and codimension is listed in Table 1 of Appendix A.

Degenerations of graded rings Of course, the Hilbert series of a graded ring does not determine that ring. Whichever method is used to compute the weights, they are not expected to match every K3 surface X, A with given Hilbert series.

Meaning 10 *If X, A is a K3 surface with genus ≤ 2, then the Hilbert series $P_{X,A}(t)$ of X will be that of some $\xi \in \mathcal{D}_{\mathrm{K3}}$. However, the weights assigned to the candidate ξ will not necessarily be those of a set of generators of the graded ring $R(X, A)$.*

Consider the well-known example of a general complete intersection of equations of degrees 2 and 4

$$Y_{2,4} \subset \mathbb{P}^4(1, 1, 1, 1, 2)$$

where the weight 2 variable does not appear in the degree 2 equation. This K3 surface has the same Hilbert series as the quartic surface in $\mathbb{P}^3$. The corresponding $\xi \in \mathcal{D}_{\mathrm{K3}}$ (which is listed even though $g = 3$) is assigned weights $(1, 1, 1, 1)$, the weights of a typical example in $\mathbb{P}^3$, rather than the weights of $\mathbb{P}^4$ above.

In [Brown, 2005], a computer search with Magma found other degenerations of K3 surfaces in codimension 1 and 2. We reproduce some results of that search in Table 4 in Appendix A as an illustration, but see [Brown, 2005] for more and for the combination of degeneration and unprojection calculations behind them.

2.3 Elliptic fibrations and Shimada's classification

By Theorem 8, the K3 database contains all K3 surfaces (at least of genus ≤ 2). From now on, we consider the converse: which candidates in the database actually arise as K3 surfaces. We understand a two different positive answers.

Definition 11 *Let $\xi \in \mathcal{D}_{\text{K3}}$ be a numerical K3 candidate from the K3 database and $(a_0, \dots, a_N)$ its weights. We say that*

(a) ξ represents a K3 Hilbert series *if there is a polarised K3 surface X, A with $P_X(t) = P_\xi(t)$.*

(b) ξ represents a K3 surface *if there is a polarised K3 surface X, A with $P_X(t) = P_\xi(t)$ whose graded ring $R(X, A)$ has a generating set $x_0, \dots, x_N \in R(X, A)$ which are homogeneous of degrees* $\deg x_i = a_i$.

We consider the stronger statement (b) in section 3 below. The natural approach to (a) is to apply the Torelli theorem for K3 surfaces; we do not do that here, although it is a straightforward computer calculation to confirm that all candidates in codimension up to 6 do at least represent a K3 Hilbert series. Instead, we compare our database with a classification of elliptic K3 surfaces due to Shimada [2000].

An elliptic K3 surface is a fibration $f\colon Y \to \mathbb{P}^1$ where Y is a nonsingular K3 surface and the general fibre is a curve of genus 1. (We do not assume that f has a section.) Shimada [2000] classifies the collections of singular fibres that do appear on elliptic K3 surfaces into 3937 different collections (many of which can appear in fibrations having different numbers of sections).

A nonexistence result There are candidates that cannot easily represent a K3 Hilbert series. A polarisation A is said to be *simple* if it intersects the exceptional locus of each singularity transversely at a single point. The candidate in the theorem below is number 76 (of genus 0) in $\mathcal{D}_{\text{K3}}$.

Theorem 12 *Let $\xi = (g, \mathcal{B}) \in \mathcal{D}_{\text{K3}}$ be the candidate with $g = 0$ and $\mathcal{B} = \{\frac{1}{2}(1,1), 2 \times \frac{1}{10}(1,9)\}$. Then there does not exist a polarised K3 surface X, A with $P_X(t) = P_\xi(t)$ for which the polarisation is simple.*

Proof Suppose X, A is a polarised K3 surface with $P_X(t) = P_\xi(t)$. We have $H^0(X, A) = 1$ since $g = 0$, so we may regard $A \subset X$ as an effective divisor with $A^2 = -2$. Let $\varphi\colon Y \to X$ be the minimal resolution of singlarities; Y is a nonsingular K3 surface. We estimate the rank of the Picard group $\operatorname{Pic}(Y)$.

In the group $\operatorname{Pic}(Y)$, the set of exceptional curves of φ are independent of each other and of the components of A. The exceptional curves will generate a subgroup of rank 19 if the singularities of X are exactly those of the basket. The list of possible degenerations of a basket in display (4) shows that that if the singularities of X are not those of the basket, then the exceptional curves will generate a subgroup of rank at least 20. Since the Picard rank of a K3 surface is at most 20, we conclude that the singularities of X are those of its basket and that A is an irreducible rational curve—otherwise its components would also contribute independently to a rank exceeding 20.

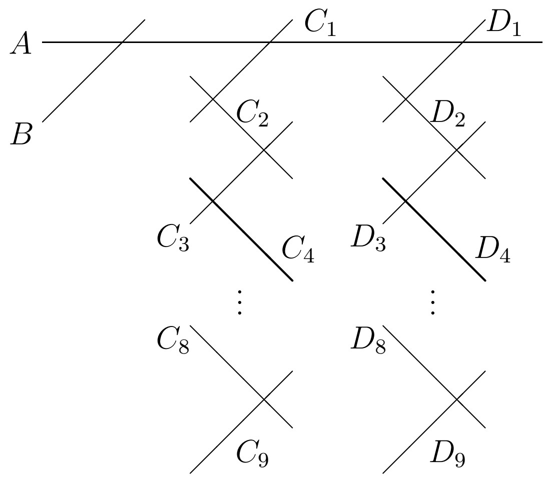


Figure 1: A configuration of curves on Y

So, since the polarisation is simple, the configuration of 20 nonsingular rational curves, each with selfintersection -2, pictured in Figure 1 lies on Y. The divisor

$$2B + 4A + 3C_1 + 2C_2 + C_3 + 3D_1 + 2D_2 + D_3$$

is an elliptic fibre $\widetilde{\mathrm{E}}_7$ on Y. This fibre generates an elliptic fibration on Y with at least 2 sections (being the two exceptional curves C_4, D_4 adjacent to the $\widetilde{\mathrm{E}}_7$ configuration). The remaining exceptional curves must be contained in other elliptic fibres. Thus the only possibilities for the singular fibres of this fibration are $\widetilde{\mathrm{E}}_7 + \widetilde{\mathrm{A}}_5 + \widetilde{\mathrm{A}}_5$, $\widetilde{\mathrm{E}}_7 + \widetilde{\mathrm{A}}_5 + \widetilde{\mathrm{A}}_5 + \widetilde{\mathrm{A}}_1$ and $\widetilde{\mathrm{E}}_7 + \widetilde{\mathrm{A}}_{11}$. Such combinations of elliptic fibres do occur according to Shimada's classification [Shimada, 2000], but they only occur with no sections at all or with exactly 1 section. So Y cannot exist as a K3 surface, and so neither does X. Q.E.D.

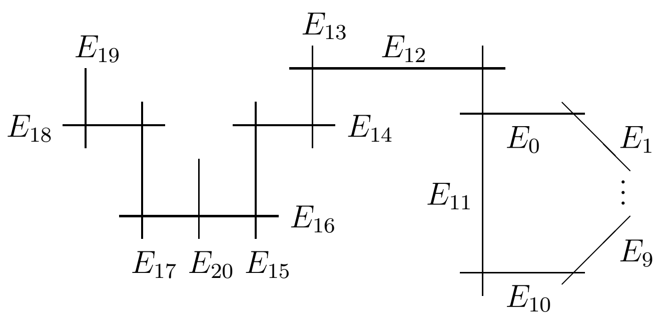
Realising a Hilbert series In a closely-related example, let $f\colon Y \to \mathbb{P}^1$ be the elliptic K3 surface number 3305 in Shimada's classification. It has two singular fibres, of types $\widetilde{\mathrm{E}}_7$ and $\widetilde{\mathrm{A}}_{11}$ respectively, and was one of the cases considered in the proof of Theorem 12. Furthermore, the Mordell–Weil group of f contains exactly one element which is the unique section of f. Therefore, the K3 surface Y contains the configuration of -2-curves pictured in Figure 2.

We define a $\mathbb{Q}$-divisor B on Y supported on this configuration of curves:

$$B = \frac{1}{16}(E_1 + 2E_2 + \cdots + 15E_{15}) + E_{16} + \frac{1}{4}(3E_{17} + 2E_{18} + E_{19}) + \frac{1}{2}E_{20}.$$

It is easy to check that B is $\mathbb{Q}$-ample (modulo some -2-curves in its support on which it is trivial) and that some multiple of B gives a morphism φ of Y to some projective space that is birational to its image and contracts all of the curves E_i of the configuration except E_0 and E_{16}. Let $X = \varphi(Y)$ and define A to be the (integral) divisor $\varphi_*(B)$ on X. Again, it is easy to check that $A^2 = 3/16$ and that the basket $\mathcal{B}$ and genus g of X, A are

$$\mathcal{B} = \left\{\frac{1}{2}(1,1), \frac{1}{4}(1,3), \frac{1}{16}(1,15)\right\} \quad \text{and} \quad g = 0. \tag{5}$$



Section of the fibration is E_{12}

Fibre $\widetilde{\mathrm{E}}_7$ is $(E_{13} + E_{19}) + 2(E_{14} + E_{18} + E_{20}) + 3(E_{15} + E_{17}) + 4E_{16}$

Fibre $\widetilde{\mathrm{A}}_{11}$ is $E_0 + E_1 + \cdots + E_{11}$

Figure 2: Shimada's elliptic fibration number 3305

Indeed, $\mathcal{D}_{\mathrm{K3}}$ contains such a numerical K3 candidate $\xi = (g, \mathcal{B})$. In the K3 database it is number 35 (of genus 0) where it is described provocatively as

$$X \subset \mathbb{P}^{12}(1, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16).$$

We conclude that ξ represents a K3 Hilbert series. But bear in mind what we are *not* claiming: while this description is meaningful, one cannot conclude that there really is a quasismooth K3 surface in this wps that realises ξ.

3 Numerical unprojection and weights

Theorem 12 shows that there are candidates in the database that might not even represent a K3 Hilbert series, let alone a K3 surface. Even so, here we attempt to compute plausible weights for every $\xi \in \mathcal{D}_{\mathrm{K3}}$ as a first step towards the stronger statement (b).

The results of Reid, Fletcher [2000] and Altınok [1998] imply that every $\xi \in \mathcal{D}_{\mathrm{K3}}$ with codimension at most 3 represents a K3 surface. Altınok also uses unprojection methods to show that the majority of ξ with codimension 4 represent a K3 surface, which Frantzen [2004] extends to confirm the same for some in codimension 5. We use unprojection methods to make predictions of weights in higher codimension—in the precise sense of Computation 17 below—although we cannot carry out the calculations to confirm that every $\xi \in \mathcal{D}_{\mathrm{K3}}$ represents a K3 surface.

3.1 Type I and Type II_n unprojections

Kustin–Miller unprojections Type I, or Kustin–Miller, unprojection is a general operation that constructs bigger Gorenstein rings from smaller ones. We use it to mean the map $X \dashrightarrow Y$ in the following theorem.

Theorem 13 (Papadakis–Reid [2004]) *Let X, A be a polarised K3 surface and $X \subset \mathbb{P}(a_0, \dots, a_N)$ its embedding by A. Suppose that X contains the coordinate line $C = \mathbb{P}(a_i, a_j)$ and that X is quasismooth along C in this embedding. Then*

(a) There is a K3 surface $Y \subset \mathbb{P}(a_0, \dots, a_N, a_i + a_j)$ containing the coordinate point $P_{N+1} = (0, \dots, 0, 1)$.

(b) The Gorenstein projection of Y from P_{N+1} is a birational map $Y \dashrightarrow X$ with exceptional sets $C \subset X$ and $P_{N+1} \in Y$. The birational inverse $X \to Y$ is the contraction of the -2-curve $C \subset X$.

(c) If $a_i + a_j + a_k > a_\ell$ for every $k, \ell \in \{0, \dots, N\} \setminus \{i, j\}$, then the unprojection equations of Y embedded in $\mathbb{P}(a_0, \dots, a_N, a_i + a_j)$ have no linear terms.

(d) Y is polarised by $B = A + \frac{1}{a_i+a_j}C$, and its Hilbert series is

$$P_{Y,B}(t) = P_{X,A}(t) + \frac{t^{a_i+a_j}}{(1-t^{a_i+a_j})(1-t^{a_i})(1-t^{a_j})}.$$

In particular, the genus of Y, B equals the genus of X, A.

(e) Let $\mathcal{B}_Y$ be the basket of Y and $\mathcal{B}_X$ that of X. Then

$$\mathcal{B}_X \cup \left\{ \frac{1}{a_i+a_j}(a_i, a_j) \right\} = \mathcal{B}_Y \cup \left\{ \frac{1}{a_i}(a_j, -a_j), \frac{1}{a_j}(a_i, -a_i), \right\}$$

where any singularity type $\frac{1}{r}(a, -a)$ of index $r = 1$ can be omitted.

The proof of most of this theorem is given (or implicit) across a number of sources including [Papadakis and Reid, 2004; Altınok, 1998; Altınok *et al.*, 2002; Frantzen, 2004], so we sketch a proof here for convenience using only the main theorem of [Papadakis and Reid, 2004].

Proof The setup $C \subset X$ is of a Type I unprojection: the ideals of $X \subset \mathbb{P}^N$ and $C \subset \mathbb{P}^N$ are Gorenstein. It follows by [Papadakis and Reid, 2004] Theorem 1 that there is a rational form $s \in \mathbb{C}(\overline{X})$ on the affine cone $\overline{X}$ of X, unique up to scalar multiple and of weight $a_i + a_j$, that has a single pole along C. The extension of algebras $R(X, A) \subset R(X, A)[s]$ inside $\mathbb{C}(\overline{X})$ defines a birational map of varieties $X \dashrightarrow Y$ contracting exactly C. Since X is quasismooth, this map is a morphism and so it is the contraction of a -2-curve on X, and X is a partial resolution of Y. In particular, the resulting Y is again a K3 surface.

If $x_0, \dots, x_N$ are given generators of $R(X, A)$ of degrees $\deg x_i = a_i$, we extend this list by $x_{N+1} = s$ to give an embedding of Y as in (a). Eliminating s from the coordinate ring of Y in this embedding recovers X, which is (b).

According to the recipe of [Papadakis and Reid, 2004] Theorem 1, the unprojection equations are of the form

$$x_{N+1}x_k = g_k \quad \text{for } k \neq i, j \text{ and some } g_k \in R(X, A).$$

These equations have degree $a_i+a_j+a_k$ for k in the set $\{0, \dots, N\} \setminus \{i, j\}$. The condition given in (c) is that this degree is higher than any of the variables, and so these variables cannot appear linearly.

The divisor of s on X contains C with coefficient -1 and is linearly equivalent to $(a_i + a_j)A$. So expressed as a divisor on X before contracting

C, the hyperplane section of Y is B as stated in (d). The formula computes the number of monomials added to $R(X, A)$ in each degree by the inclusion of s. It counts multiples of s by the variables x_i, x_j and by s itself—all other monomials sx_k can be eliminated by the unprojection equations.

Finally, (e) follows from the formula in (d) together with Theorem 2. Alternatively, one sees from the unprojection equations that $P_{N+1} \in Y$ is of type $\frac{1}{a_i+a_j}(a_i, a_j)$ which forces the quotient singularities of X along C to be the stated pair. Q.E.D.

In principle, this theorem provides an inductive framework for generating K3 surfaces in high codimension to realise items in $\mathcal{D}_{\mathrm{K3}}$. Indeed, this is how Altınok [1998] and Frantzen [2004] construct K3 surfaces in codimension 4 and 5. However, verifying that such $X \supset C$ exist and is quasismooth seems difficult in general. Instead, we use the theorem to generate plausible weights for any $\xi \in \mathcal{D}_{\mathrm{K3}}$ as follows.

Definition 14 *For $\eta = (g, \mathcal{B}) \in \mathcal{D}_{\mathrm{K3}}$ and $p = \frac{1}{r}(a, -a) \in \mathcal{B}$, the* numerical projection of p in η *is $\xi = (g, \mathcal{B}')$ where*

$$\mathcal{B}' \cup \left\{ \frac{1}{r}(a, -a) \right\} = \mathcal{B} \cup \left\{ \frac{1}{a}(r, -r), \frac{1}{r-a}(r, -r) \right\}$$

and we omit any singularity of the type $\frac{1}{s}(c, -c)$ of index $s = 1$.

Algorithm 15 (Type I forcing) *For fixed genus $-1 \le g \le 2$, let $\mathcal{D}_{\mathrm{K3}}(g)$ be the subset of $\mathcal{D}_{\mathrm{K3}}$ of numerical K3 candidates with genus g ordered in increasing Hilbert series order.*

For $\eta = (g, \mathcal{B}) \in \mathcal{D}_{\mathrm{K3}}(g)$ do

(1) if P_η is that of a known K3 surface of codimension ≤ 2, then assign it these known weights; continue with the next η.

(2) for each $p = \frac{1}{r}(a, -a) \in \mathcal{B}$

(a) compute the numerical projection ξ of p in η

(b) if $(g, \xi) \in \mathcal{D}_{\mathrm{K3}}(g)$ and the pair $a, r-a$ occurs among the weights W of ξ, then let the weights of ξ be $W \cup \{r\}$; continue with the next η.

(3) apply the naive algorithm of section 2.2 to generate weights for η; continue with the next η.

In (1), we take the $95+84$ K3 surfaces in codimension at most 2 as known. Of course, in (2)(b) the pair (g,ξ) will be in $\mathcal{D}_{\mathrm{K3}}$ if and only if $A_\xi^2>0$, and furthermore the weights of ξ will be known inductively because ξ appears in $\mathcal{D}_{\mathrm{K3}}(g)$ ahead of η by the Hilbert series order together with Theorem 13(d).

Part (3) of the algorithm is not very satisfactory. One can improve on the naive algorithm by adding other weights to realise the basket better, for instance. But we don't discuss that here since in practice the unprojection step (2) is enough once higher unprojections are included.

Higher unprojections Type I unprojections are only one kind of unprojection calculation associated to Gorenstein rings. At the time of writing, it is the only one for which the theory is complete, although Papadakis [2005] has recently proved some results for Type II. However, it is still possible to calculate with other kinds. Experience from examples suggests the following numerical characterisation of another type of unprojection.

Definition 16 *Suppose $\xi\in\mathcal{D}_{\mathrm{K3}}$ is a numerical projection of η—that is, ξ and η are related as in Definition 14. Let W_ξ be the weights of ξ and n be a positive integer. Then the projection is a* numerical Type II_n projection *if $(r-a)\in W_\xi$ and $n+1$ is the smallest positive integer k for which $ka\in W_\xi\setminus\{r-a\}$ (or the analogous statement with a and $r-a$ switching roles).*

In this situation and notation, we define the expected weights of η *to be*

$$W_\eta=W_\xi\cup\{r,r+a,r+2a,\cdots,r+na\}.$$

The idea is that the new weight r corresponds to an unprojection variable s as for Type I, but that additional variables are needed to make the unprojection projectively normal.

For example, projecting from

$$\frac{1}{2}(1,1)\in Y\subset\mathbb{P}^5(1,2,2,3,5,7)$$

is a numerical Type II_1 projection and has image

$$\mathbb{P}^1\subset X_{15}\subset\mathbb{P}^5(1,2,5,7).$$

A single variable of weight 2 has been eliminated, and with it the variable of weight 3 that polarised the singularity is also eliminated. One could eliminate the weight 3 variable alone to see the non-normal unprojection. See section 4 for this projection in the database.

We can force Type II unprojections just as for Type I in Algorithm 15. In constructing $\mathcal{D}_{\mathrm{K3}}$, exactly this is done using Types II_1 and II_2 once the possibility of a Type I unprojection has been exhausted. Once constructed, the whole K3 database is subjected to the following consistency check.

Computation 17 (Numerical Type I and II consistency) *The weights of numerical K3 candidates in $\mathcal{D}_{\mathrm{K3}}$ are consistent with all projections of numerical Type I and Type II_n for any $n > 0$ from any candidate of codimension 3 or more.*

The proof is a computer calculation: for each $\xi \in \mathcal{D}_{\mathrm{K3}}$ that does not correspond to a known K3 surface in codimension 1 or 2, the weights of ξ are computed according to every projection of numerical Types I and II, and the results are required to be the same. Candidates in codimension 1 or 2 are ignored since they are already known to be correct (and projection can be more complicated in such small graded rings).

This computation is important. Together with the few hundred initial cases, it is the main supporting evidence that the families described by the database do represent K3 surfaces in the sense of Definition 11(b).

Meaning 18 *The weights associated to numerical K3 candidates in $\mathcal{D}_{\mathrm{K3}}$ are consistent with the existence of Type I and II unprojections between K3 surfaces realising them.*

One could use the same unprojection calculus to predict weights for any $(g, \mathcal{B})$ with $g \geq 3$ which is not listed in $\mathcal{D}_{\mathrm{K3}}$. The results would be the same as those for $(2, \mathcal{B})$ but with the inclusion of the weight 1 an additional $g-2$ times. Such continuation of $\mathcal{D}_{\mathrm{K3}}$ would be visible in Table 1 as the $g = 2$ column copied in each column to the right, but in higher codimension at each higher genus (with the two candidates in minimal codimension put in codimensions $g-1$ and $g-2$, as at the head of the $g = 3$ column).

3.2 K3 surfaces admitting no Gorenstein projection

Since the existence of projections is the basis for the computation of higher codimension weights in $\mathcal{D}_{\mathrm{K3}}$, the following result limiting those numerical K3 candidates having no projections, or only projections not of numerical Type I or II, is important.

Computation 19 *Let $\xi = (g, \mathcal{B}) \in \mathcal{D}_{\mathrm{K3}}$ be a numerical K3 candidate.*

If $g \le 2$ and ξ does not have any numerical projection to another polarised K3 surface, then ξ is one of the following:

g	ξ
-1	$\operatorname{codim}\xi \le 4$ *and ξ is one of 36 cases of Table 2*
0	$\operatorname{codim}\xi = 1$ *and ξ is one of 6 cases of Table 3*
1	*numerical data of* $X_{12} \subset \mathbb{P}(1,1,4,6)$
2	*numerical data of* $X_6 \subset \mathbb{P}(1,1,1,3)$

If ξ has at least one numerical projection but does not have a numerical projection of Type I or II, then $g = -1$ and ξ is of the form

$$X \subset \mathbb{P}^{7+3k}(2^{4+k}, 3^{4+2k}) \quad \text{for } k = 0, \ldots, 6$$

(where 2^n indicates n occurrences of weight 2) with $\mathcal{B} = \{(10+k) \times \frac{1}{2}(1,1)\}$ and $A^2 = (k+2)/2$, or ξ is one of the following 3 cases:

$$\begin{array}{ll} X \subset \mathbb{P}^8(8,8,9,10,11,12,13,14,15), & A^2 = \frac{1}{2} + \frac{3}{4} + \frac{3\cdot 5}{8} + \frac{8}{9} = 1/72; \\ X \subset \mathbb{P}^7(7,7,8,9,10,11,12,13), & A^2 = \frac{2\cdot 5}{7} + \frac{3\cdot 4}{7} + \frac{7}{8} = 1/56; \\ X \subset \mathbb{P}^5(4,5,5,6,7,8), & A^2 = \frac{1}{2} + \frac{3}{4} + 2 \times \frac{4}{5} + \frac{2\cdot 3}{5} = 1/20 \end{array}$$

where $\frac{1}{r}(a,-a) \in \mathcal{B}$ is represented by its contribution $b(r-b)/r$ to A^2 in (3).

The first half of this computation already appeared in [Brown, 2003]. The second half can be computed from the K3 database using code similar to that of section 4.

4 Using the K3 database

There are three ways to access the K3 database. The main one is to use Magma as described below. Second, the website [Brown *et al.*, 2004] has a bureaucratic front-end to Magma with a form to fill in that can be submitted to the K3 database. And third, there is a SQL-style version of the database posted on the website [Brown *et al.*, 2004] that can be downloaded and installed under an SQL server—of course, this is static and does not include some data that is computed live by Magma.

The K3 database in MAGMA The computer algebra system MAGMA [Cannon, 2005; Bosma *et al.*, 1997] (version 2.11 or higher) contains a database of 24,099 representative K3 surfaces. We give an example of a continuous session using this database. Having already started MAGMA (typically by typing `magma` at a command line), we name the K3 database D.

```
> D := K3Database();
> D;
The database of K3 surfaces
```

Now we pick out a surface X with given weights. It can be analysed using various function calls like `Degree(X)`. But simply printing it on screen presents all of its useful data.

```
> X := K3Surface(D,[1,2,2,3,5,7]);
> Degree(X);
5/7
> X;
K3 surface no.797, genus 0, in codimension 3 with data
  Weights: [ 1, 2, 2, 3, 5, 7 ]
  Basket: 2 x 1/2(1,1), 1/7(2,5)
  Degree: 5/7           Singular rank: 8
  Numerator: -t^20 + ... + t^11 - t^9 - t^8 - t^7 - t^6 + 1
  Projection to codim 2 K3 no.796 -- type I from 1/7(2,5)
  Projection to codim 1 K3 no.251 -- type II_1 from 1/2(1,1)
  Unproj'n from codim 4 K3 no.798 -- type I from 1/9(2,7)
  Unproj'n from codim 4 K3 no.816 -- type I from 1/3(1,2)
  Unproj'n from codim 5 K3 no.1642 -- type II_1 from 1/2(1,1)
```

In this case, reading the numerator suggests there are four equations of weights 6,7,8,9 respectively. In fact, it is known that Gorenstein rings in codimension 3 have an odd number of relations, so one guesses that there are 5 equations, the missing one being in degree 10, masked in the numerator by some syzygy also of degree 10. This now works, and one can write X as the five maximal pfaffians of a skew 5×5 matrix—compare with [Altınok *et al.*, 2002] Example 3.7.

The weights of X can be deduced using a Type I unprojection. Indeed, we see that the $g = 0$ surface number 796 has the right numerical properties to be a Type I projection from the $\frac{1}{7}(2,5)$ point of X.

```
> K3Surface(D,0,796);
K3 surface no.796, genus 0, in codimension 2 with data
  Weights: [ 1, 2, 2, 3, 5 ]
  Basket: 3 x 1/2(1,1), 1/5(2,3)
  Degree: 7/10          Singular rank: 7
  Numerator: t^13 - t^7 - t^6 + 1
   [ ... 2 projections and 3 unprojections including ... ]
  Unproj'n from codim 3 K3 no.797 -- type I from 1/7(2,5)
```

Indeed, this can be realised and the unprojection can be calculated—compare again [Altınok *et al.*, 2002] Example 3.7.

The surface X has a second projection. By Computation 17, the weights of X can be calculated using this projection instead and should give the same result. Indeed, looking at the image of the numerical projection, we see that its weights differ only by the missing pair $2, 3$, which is also the prediction using the numerical Type II_1 projection of Definition 16.

```
> K3Surface(D,0,251);
K3 surface no.251, genus 0, in codimension 1 with data
  Weights: [ 1, 2, 5, 7 ]
  Basket: 1/2(1,1), 1/7(2,5)
  Degree: 3/14          Singular rank: 7
  Numerator: -t^15 + 1
   [ ... 1 projection and 3 unprojections including ... ]
  Unproj'n from codim 3 K3 no.797 -- type II_1 from 1/2(1,1)
```

One can make more serious searches, testing predicates on each surface in the database. For example, we make a sequence containing those codimension 5 surfaces with no unprojections; there is only one of them.

```
> K3s := [ X : X in D | Codimension(X) eq 5 and
                            #Unprojections(X) eq 0 ];
> #K3s;
1
> Y := K3s[1];
> Weights(Y);
[ 7, 7, 8, 9, 10, 11, 12, 13 ]
> Basket(Y);
1/7(2,5), 1/7(3,4), 1/8(1,7)
```

Or we can confirm Computation 1; $X_{66} \subset \mathbb{P}(5,6,22,33)$ has degree 1/330.

```
> [ Weights(X) : X in D | Degree(X) le 1/330 ];
[ [ 5, 6, 22, 33 ] ]
```

To give a typical calculation, we write a function to list all projections of the K3 surface number 35 constructed in section 2.3.

```
> function projections(X,D)
>     P := {X};
>     todo := {X};
>     repeat
>         todo := &join[ { K3Surface(D,Genus(X),P[1])
>                 : P in Projections(Y) } : Y in todo ];
>         P join:= todo;
>     until #todo eq 0;
>     return P;
> end function;
```

We apply this function to X and look at the codimensions of its projections.

```
> X := K3Surface(D,0,35);
> {* Codimension(Y): Y in projections(X,D) *};
{* 1^^6, 2^^2, 3^^2, 4^^2, 5^^2, 6^^2, 7^^2, 8^^2, 9, 10 *}
```

In other words,

Computation 20 *Let $\xi = (g, \mathcal{B}) \in \mathcal{D}_{\mathrm{K3}}$ be the numerical K3 candidate of display (5) in section 2.3. The set of projections of ξ consists of 22 candidates in $\mathcal{D}_{\mathrm{K3}}$ whose weights are spread across codimensions 1 to 10 as follows:*

codim	1	2	3	4	5	6	7	8	9	10
number	6	2	2	2	2	2	2	2	1	1

Since ξ represents a K3 Hilbert series, so does each of these 22 candidates.

A Tables of results

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	−1	0	1	2	3	4	5	6	7	8	9	total
1	54	32	6	2	1	0	0	0	0	0	0	95
2	45	29	6	2	1	1	0	0	0	0	0	84
3	26	29	8	3	2	1	1	0	0	0	0	70
4	60	54	15	6	3	2	1	1	0	0	0	142
5	58	63	21	8	6	3	2	1	1	0	0	163
6	80	98	35	15	8	6	3	2	1	1	0	249
7	81	116	49	21	15	8	6	3	2	1	1	303
8	128	182	79	35								
9	107	208	109	49								
10	192	312	171	79								
11	167	369	236	109								
12	238	497	353	171								
13	245	603	488	236								
14	346	759	720	353								
15	316	728	982	488								
16	402	744	1419	720								
17	337	581	1930	982								
18	350	457	0	1419								
19	266	267	0	1930								
20	258	171	0	0								
21	161	85	0	0								
22	139	55	0	0								
23	93	24	0	0								
24	57	13	0	0								
25	35	3	0	0								
26	22	0	0	0								
27	12	0	0	0								
28	5	0	0	0								
29	1	0	0	0								
total	4281	6479	6627	6628								

Table 1: Number of $\xi \in \mathcal{D}_{\mathrm{K3}}$ by codimension (down) and genus (across)

K3 surface	ρ_X	Basket $\mathcal{B}$	Degree A^2
$X_{50} \subset \mathbb{P}(7,8,10,25)$	19	$\frac{1}{2}+\frac{2\cdot3}{5}+\frac{2\cdot5}{7}+\frac{7}{8}$	$1/280$
$X_{36} \subset \mathbb{P}(7,8,9,12)$	19	$\frac{2}{3}+\frac{3}{4}+\frac{3\cdot4}{7}+\frac{7}{8}$	$1/168$
$X_{40} \subset \mathbb{P}(5,7,8,20)$	18	$\frac{3}{4}+2\times\frac{2\cdot3}{5}+\frac{6}{7}$	$1/140$
$X_{66} \subset \mathbb{P}(5,6,22,33)$	18	$\frac{1}{2}+\frac{2}{3}+\frac{2\cdot3}{5}+\frac{2\cdot9}{11}$	$1/330$
$X_{38} \subset \mathbb{P}(5,6,8,19)$	18	$\frac{1}{2}+\frac{4}{5}+\frac{5}{6}+\frac{3\cdot5}{8}$	$1/120$
$X_{27} \subset \mathbb{P}(5,6,7,9)$	18	$\frac{2}{3}+\frac{4}{5}+\frac{5}{6}+\frac{3\cdot4}{7}$	$1/70$
$X_{34} \subset \mathbb{P}(4,6,7,17)$	17	$2\times\frac{1}{2}+\frac{3}{4}+\frac{5}{6}+\frac{3\cdot4}{7}$	$1/84$
$X_{54} \subset \mathbb{P}(4,5,18,27)$	17	$\frac{1}{2}+\frac{3}{4}+\frac{2\cdot3}{5}+\frac{2\cdot7}{9}$	$1/180$
$X_{32} \subset \mathbb{P}(4,5,7,16)$	17	$2\times\frac{3}{4}+\frac{4}{5}+\frac{3\cdot4}{7}$	$1/70$
$X_{30} \subset \mathbb{P}(4,5,6,15)$	16	$2\times\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+2\times\frac{4}{5}$	$1/60$
$X_{24} \subset \mathbb{P}(3,6,7,8)$	16	$\frac{1}{2}+4\times\frac{2}{3}+\frac{6}{7}$	$1/42$
$X_{48} \subset \mathbb{P}(3,5,16,24)$	16	$2\times\frac{2}{3}+\frac{4}{5}+\frac{3\cdot5}{8}$	$1/120$
$X_{21} \subset \mathbb{P}(3,5,6,7)$	16	$3\times\frac{2}{3}+\frac{2\cdot3}{5}23+\frac{5}{6}$	$1/30$
$X_{42} \subset \mathbb{P}(3,4,14,21)$	15	$\frac{1}{2}+2\times\frac{2}{3}+\frac{3}{4}+\frac{2\cdot5}{7}$	$1/84$
$X_{24} \subset \mathbb{P}(3,4,5,12)$	15	$2\times\frac{2}{3}+2\times\frac{3}{4}+\frac{2\cdot3}{5}$	$1/30$
$X_{18} \subset \mathbb{P}(3,4,5,6)$	15	$\frac{1}{2}+3\times\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$	$1/20$
$X_{15} \subset \mathbb{P}(3,3,4,5)$	14	$5\times\frac{2}{3}+\frac{3}{4}$	$1/12$
$X_{30} \subset \mathbb{P}(2,6,7,15)$	14	$5\times\frac{1}{2}+\frac{2}{3}+\frac{6}{7}$	$1/42$
$X_{42} \subset \mathbb{P}(2,5,14,21)$	14	$3\times\frac{1}{2}+\frac{4}{5}+\frac{3\cdot4}{7}$	$1/70$
$X_{26} \subset \mathbb{P}(2,5,6,13)$	14	$4\times\frac{1}{2}+\frac{2\cdot3}{5}+\frac{5}{6}$	$1/30$
$X_{22} \subset \mathbb{P}(2,4,5,11)$	13	$5\times\frac{1}{2}+\frac{3}{4}+\frac{4}{5}$	$1/20$
$X_{30} \subset \mathbb{P}(2,3,10,15)$	12	$3\times\frac{1}{2}+2\times\frac{2}{3}+\frac{2\cdot3}{5}$	$1/30$
$X_{18} \subset \mathbb{P}(2,3,4,9)$	12	$4\times\frac{1}{2}+2\times\frac{2}{3}+\frac{3}{4}$	$1/12$
$X_{12} \subset \mathbb{P}(2,3,3,4)$	12	$3\times\frac{1}{2}+4\times\frac{2}{3}$	$1/6$
$X_{14} \subset \mathbb{P}(2,2,3,7)$	10	$7\times\frac{1}{2}+\frac{2}{3}$	$1/6$
$X_{24,30} \subset \mathbb{P}(8,9,10,12,15)$	19	$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{2\cdot3}{5}+\frac{8}{9}$	$1/180$
$X_{18,30} \subset \mathbb{P}(6,8,9,10,15)$	18	$2\times\frac{1}{2}+2\times\frac{2}{3}+\frac{4}{5}+\frac{7}{8}$	$1/120$
$X_{16,18} \subset \mathbb{P}(4,6,7,8,9)$	17	$2\times\frac{1}{2}+\frac{2}{3}+2\times\frac{3}{4}+\frac{6}{7}$	$1/42$
$X_{14,16} \subset \mathbb{P}(4,5,6,7,8)$	17	$\frac{1}{2}+2\times\frac{3}{4}+\frac{2\cdot3}{5}+\frac{5}{6}$	$1/30$
$X_{12,14} \subset \mathbb{P}(4,4,5,6,7)$	16	$2\times\frac{1}{2}+3\times\frac{3}{4}+\frac{4}{5}$	$1/20$
$X_{10,12} \subset \mathbb{P}(3,4,4,5,6)$	15	$\frac{1}{2}+2\times\frac{2}{3}+3\times\frac{3}{4}$	$1/12$
$X_{6,6} \subset \mathbb{P}(2,2,2,3,3)$	10	$9\times\frac{1}{2}$	$1/2$
$X_{16,\dots,20} \subset \mathbb{P}(5,6,7,8,9,10)$	18	$\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{2\cdot3}{5}+\frac{6}{7}$	$1/42$
$X_{14,\dots,18} \subset \mathbb{P}(5,5,6,7,8,9)$	18	$\frac{4}{5}+2\times\frac{2\cdot3}{5}+\frac{5}{6}$	$1/30$
$X \subset \mathbb{P}(6,6,7,8,9,10,11)$	18	$2\times\frac{1}{2}+2\times\frac{2}{3}+\frac{5}{6}+\frac{6}{7}$	$1/42$
$X \subset \mathbb{P}(5,6,6,7,8,9,10)$	18	$\frac{1}{2}+\frac{2}{3}+\frac{2\cdot3}{5}+2\times\frac{5}{6}$	$1/30$

Table 2: K3 surfaces with $g=-1$ having no Gorenstein projection

K3 surface	ρ_X	Basket $\mathcal{B}$	Degree A^2
$X_{42} \subset \mathbb{P}(1,6,14,21)$	10	$\frac{1}{2}+\frac{2}{3}+\frac{6}{7}$	$1/42$
$X_{36} \subset \mathbb{P}(1,5,12,18)$	10	$\frac{2\cdot 3}{5}+\frac{5}{6}$	$1/30$
$X_{30} \subset \mathbb{P}(1,4,10,15)$	9	$\frac{1}{2}+\frac{3}{4}+\frac{4}{5}$	$1/20$
$X_{24} \subset \mathbb{P}(1,3,8,12)$	8	$2\times\frac{2}{3}+\frac{3}{4}$	$1/12$
$X_{18} \subset \mathbb{P}(1,2,6,9)$	6	$3\times\frac{1}{2}+\frac{2}{3}$	$1/6$
$X_{10} \subset \mathbb{P}(1,2,2,5)$	6	$5\times\frac{1}{2}$	$1/2$

Table 3: K3 surfaces with $g = 0$ having no Gorenstein projection

general member	$\rightsquigarrow$	degeneration
$X_4 \subset \mathbb{P}(1,1,1,1)$	$\rightsquigarrow$	$Y_{2,4} \subset \mathbb{P}(1,1,1,1,2)$
$X_6 \subset \mathbb{P}(1,1,2,2)$	$\rightsquigarrow$	$Y_{3,6} \subset \mathbb{P}(1,1,2,2,3)$
$X_8 \subset \mathbb{P}(1,2,2,3)$	$\rightsquigarrow$	$Y_{4,8} \subset \mathbb{P}(1,2,2,3,4)$
$X_{10} \subset \mathbb{P}(1,2,3,4)$	$\rightsquigarrow$	$Y_{5,10} \subset \mathbb{P}(1,2,3,4,5)$
$X_{12} \subset \mathbb{P}(1,2,4,5)$	$\rightsquigarrow$	$Y_{6,12} \subset \mathbb{P}(1,2,4,5,6)$
$X_{12} \subset \mathbb{P}(2,3,3,4)$	$\rightsquigarrow$	$Y_{6,12} \subset \mathbb{P}(2,3,3,4,6)$
$X_{14} \subset \mathbb{P}(2,3,4,5)$	$\rightsquigarrow$	$Y_{7,14} \subset \mathbb{P}(2,3,4,5,7)$
$X_{18} \subset \mathbb{P}(3,4,5,6)$	$\rightsquigarrow$	$Y_{9,18} \subset \mathbb{P}(3,4,5,6,9)$
$X_6 \subset \mathbb{P}(1,1,1,3)$	$\rightsquigarrow$	$Y_{2,6} \subset \mathbb{P}(1,1,1,2,3)$
$X_{12} \subset \mathbb{P}(1,2,3,6)$	$\rightsquigarrow$	$Y_{4,12} \subset \mathbb{P}(1,2,3,4,6)$
$X_{18} \subset \mathbb{P}(1,3,5,9)$	$\rightsquigarrow$	$Y_{6,18} \subset \mathbb{P}(1,3,5,6,9)$
$X_{18} \subset \mathbb{P}(2,3,4,9)$	$\rightsquigarrow$	$Y_{6,18} \subset \mathbb{P}(2,3,4,6,9)$
$X_{24} \subset \mathbb{P}(3,4,5,12)$	$\rightsquigarrow$	$Y_{8,24} \subset \mathbb{P}(3,4,5,8,12)$
$X_{30} \subset \mathbb{P}(4,5,6,15)$	$\rightsquigarrow$	$Y_{10,30} \subset \mathbb{P}(4,5,6,10,15)$

Table 4: Some codimension 2 degenerations among the famous 95